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A resolution of the blow-off singularity for similarity flow on a flat plate

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A study is made of uniform flow past a semi-infinite flat plate with a similarity injection distribution of boundary-layer magnitude. Attention is focused on a solution at exactly the critical injection rate for which classical boundary-layer theory predicts the blow-off singularity. Following a description of the more recent interaction analyses which also fail at the critical rate, a new theory is developed which leads to physically meaningful results. In particular, it is shown that the non-monotonic variation in wall shear with increasing injection rate near the critical value, noted by Klemp & Acrivos (1972), is real. A delicate interplay of weak pressure interactions and viscous effects is shown to be responsible for this surprising phenomenon.

1. Introduction

The boundary-layer separation phenomenon associated with mass addition from a flat plate in a uniform flow has been considered recently by Kassoy (1970, 1971) and Klemp & Acrivos (1972). These analytical investigations are concerned with the process of blow-off arising from a similarity injection distribution of boundary-layer magnitude. The earlier work by Emmons & Leigh (1954) based on classical boundary-layer theory provided a numerical description of the approach to the blow-off condition. They showed that there exists a critical injection rate at which the wall shear vanishes. At this value the shear layer is located 'infinitely' far from the wall (on the scale of the transverse boundary-layer variable). The shear-layer solution is described by Lock's (1951) mixing layer. Motivated by this work, Kassoy (1970) developed an analytical description of the same problem. He showed how the blow-off condition is approached as the injection rate approaches the critical value. It was also shown that the classical analysis becomes singular in the mathematical sense as the critical rate is approached because of the thickening of the overall flow structure beyond the classical boundary-layer magnitude.

Kassoy (1971) and Klemp & Acrivos (1972) then considered the same physical problem for injection rates beyond the critical value but within the context of injection of boundary-layer magnitude. The classical analysis was abandoned in favour of a weak interaction theory. It was shown that the thickening of the injectant layer results in a weak interaction with the external flow which produces a small favourable pressure gradient. This pressure effect helps to accelerate the injected fluid in a way which permits appropriate entrainment by the free shear layer. This theory was shown to be invalid as the critical rate was approached from above because the assumptions upon which it was based were violated. Hence, neither the classical theory, nor the interaction analysis provides a description of the flow at the critical injection rate.

Klemp & Acrivos (1972) remarked that the wall shear distribution is not a monotonic function of the injection rate. The Emmons & Leigh (1954) and Kassoy (1970) boundary-layer theories indicate that the non-dimensional wall shear, $O(Re^{-\frac{1}{2}})$, becomes vanishingly small as the injection magnitude increases towards the critical value. The implication is that the wall shear becomes $o(Re^{-\frac{1}{2}})$. Then from the interaction theory, where the wall shear is $O(Re^{-\frac{5}{8}})$, it is observed that, as the injection magnitude decreases towards the critical value from above, the wall shear again vanishes. This in turn implies that, in the neighbourhood of the critical magnitude of injection, the wall shear is $o(Re^{-\frac{5}{8}})$, a more stringent condition than that obtained from the boundary-layer theories. This somewhat unexpected non-monotonic variation apparently is caused by the relative ineffectiveness of the interaction pressure for values of injection near the critical value.

Since both theories fail at the critical rate, it would appear likely that the zero wall shear value is not physically correct. As we shall see, a new theory must be developed to provide a proper description of the phenomena. In the present work, a calculation is made for the flow at the critical rate of injection. This theory differs fundamentally from both the classical and interaction analyses although, as would be expected, it contains elements of each. It is based upon the use of several distinguished limits of the Navier–Stokes equations for the limit of large Reynolds number. The results indicate that extremely weak viscous effects and a pressure interaction combine to produce a small but finite wall shear $\tau_w = O(Re^{-1}\log Re)$. Hence, the non-monotonic shear distribution is verified.

Although the calculation in itself may appear to be somewhat academic, the consequences of the result are significant. It shows how the inclusion of a developing, weak-interaction pressure gradient precludes the appearance of the mathematical singularity at blow-off. In point of fact, when the appropriate theory is developed, a well-defined physically acceptable flow is obtained.

2. Review

In order to set the new problem in proper perspective, it is useful to review briefly the essential details of the boundary-layer and interaction theories mentioned in the introduction. The former is based on the system

$$\psi_y\psi_{xy}-\psi_x\psi_{yy} = \psi_{yyy},$$
$$\psi_y = 1 \quad \text{for} \quad x>0, \quad y\to\infty$$
$$\psi_y = 0, \quad \psi = -C(2x)^{\frac{1}{2}} \quad \text{for} \quad x>0, \quad y=0,$$

where non-dimensional boundary-layer variables have been used. The solution is sought in the limit $\alpha = C_0 - C \to 0+$, where $C_0 = 0{\cdot}87574$ is the critical injection rate. Kassoy's (1970) two-layer analysis consists first of a wall region in which

$$\psi(x,y;\alpha) \sim (2x)^{\frac{1}{2}}[-C_0+\alpha-(\alpha/\log\alpha)(e^{C_0\eta}-C_0\eta-1)+\ldots],$$

where $\eta = y(2x)^{-\frac{1}{2}}$. The first two terms represent the undisturbed injected fluid, while the third term arises from a relatively weak viscous effect which accelerates the fluid in the downstream direction. The exponential growth of the latter implies that the solution is singular for large η. The second region, essentially a free shear layer, is described by

$$\psi(x,y;\,\alpha) \sim (2x)^{\frac{1}{2}}[f_0(Z)+\ldots],$$

where f_0 is Lock's mixing-layer solution and $Z = \eta + C_0^{-1}\log\alpha + \ldots$. In the course of analysis, it is found that the magnitude of the non-dimensional wall shear is $\tau_w = \mu'[\rho'_\infty U'^2_\infty]^{-1}\,\partial u'(x',0)/\partial y' = O(Re^{-\frac{1}{2}}\alpha\log^{-1}\alpha)$, where primes denote dimensional quantities. The location of the zero (dividing) streamline, in terms of an Euler (rather than boundary-layer) variable, is given by $\delta_D = O(Re^{-\frac{1}{2}}\log\alpha)$. The latter result, in particular, indicates that the solution is not uniformly valid in the limit of large Reynolds number. An $\alpha = \alpha(Re)$ can be specified for which the boundary-layer thickness is greater than $O(Re^{-\frac{1}{2}})$. Hence, the scaling assumption used to derive the boundary-layer limit of the Navier–Stokes equation is violated. These considerations suggest that in reality $\tau_w = o(Re^{-\frac{1}{2}})$ and the overall thickness is larger than $O(Re^{-\frac{1}{2}})$ (but small compared with unity) for C sufficiently close to C_0. The latter result further suggests that an interaction between the relatively thick internal structure and the external flow will lead to weak pressure gradients that do not appear in the boundary-layer theory.

In an attempt to overcome the deficiencies of the classical theory, Kassoy (1971) and Klemp & Acrivos (1972) developed an interaction theory based on several different limiting forms of the Navier–Stokes equations. The analysis was in terms of a three-layer structure, each with distinct physical characteristics. The wall region, $O(Re^{-\frac{1}{3}})$ in thickness, is an inviscid rotational layer in which a weak interaction pressure gradient $O(Re^{-\frac{1}{3}})$ turns the injected flow. The pressure effect arises from an interaction of the relatively thick wall layer with the external flow. Above the wall layer, there appears a free shear layer described essentially by Lock's mixing-layer solution. To lowest order, the weak pressure gradient does not influence this flow.

A matching of the wall- and shear-layer solutions indicates that the former must be truncated at a finite value of the wall-layer variable. The third region is the inviscid irrotational external flow described by a uniform potential flow plus an $O(Re^{-\frac{1}{3}})$ correction caused by the effective displacement body arising from the internal structure. A slender-body analysis produces the required interaction pressure. The important results for the wall shear and dividing-streamline location are

$$\tau_w = O(Re^{-\frac{5}{6}}[1-(C_0/C)^{\frac{2}{3}}]^{\frac{1}{3}}), \quad \delta_D = O(Re^{-\frac{1}{3}}[1-(C_0/C)^{\frac{2}{3}}]^{\frac{1}{3}}).$$

The former implies that, in reality, for C sufficiently close to C_0, $\tau_w = o(Re^{-\frac{5}{6}})$ while the latter shows that the dimension of the wall layer becomes less than $O(Re^{-\frac{1}{3}})$, thus precluding the validity of the analysis. In fact, one can show that, when $C - C_0 = O(Re^{-\frac{1}{2}})$, the location of the dividing streamline in terms of the external flow variable is given by $\delta_D = O(Re^{-\frac{1}{2}})$, which is the classical boundary-layer magnitude. Hence, the interaction theory can be formally valid only for $C - C_0 \gg Re^{-\frac{1}{2}}$.

These results seem to suggest that for $C = C_0$ the internal structure has a thickness somewhere between $O(Re^{-\frac{1}{2}})$ and $O(Re^{-\frac{1}{3}})$ and that the associated wall shear is less than $O(Re^{-\frac{5}{6}})$.

3. Problem formulation

The complete mathematical system describing an incompressible uniform flow past a semi-infinite flat plate from which mass is injected with the critical boundary-layer magnitude associated with a similarity distribution may be written in the non-dimensional form

$$[\psi_y(\partial/\partial x) - \psi_x(\partial/\partial y) - \epsilon\nabla^2]\nabla^2\psi = 0, \quad \psi = \psi(x, y; \epsilon), \tag{1a}$$

$$\psi_y = 1, \quad \psi_x = 0 \quad \text{as} \quad x \to -\infty, \tag{1b}$$

$$\psi_y = 0, \quad \psi_x = -\epsilon^{\frac{1}{2}}C_0(2x)^{-\frac{1}{2}} \quad \text{for} \quad x > 0, \quad y = 0, \tag{1c}$$

where the 'injection constant' $C_0 = 0{\cdot}87574\ldots$. The variables are defined with respect to dimensional quantities by $\psi = \psi'/U'_\infty L'$, $x = x'/L'$ and $y = y'/L'$, where U'_∞ is the characteristic velocity and L' is some arbitrary length. These definitions are identical to those used by Klemp & Acrivos (1972). The parameter $\epsilon = Re^{-1} = (U'_\infty L'/\nu')^{-1}$, where ν' is the kinematic viscosity. The operator ∇^2 is the two-dimensional Laplacian.

The dimensional solution of the similarity problem posed in (1) must be independent of the arbitrary length scale L'. Hence the explicit x dependence of the solution must appear in terms of the asymptotically large group

$$x/\epsilon = U'_\infty x'/\nu'.$$

In particular a matched asymptotic expansion procedure must be developed which is based on the limit $x/\epsilon \to \infty$. The regions to be considered are a wall layer composed basically of injected fluid, a free shear layer which is strongly viscous in character and the slightly (but critically) disturbed external uniform flow.

There are two procedures which may be used to obtain the desired result. In the first, one seeks asymptotic co-ordinate expansions of the form

$$g(x, y; \epsilon) \sim \sum_{n=0}^{\infty} \alpha_{ni}(x/\epsilon) f_{ni}(\eta_i), \quad \lim_{x/\epsilon \to \infty} (\alpha_{(n+1)i}/\alpha_{ni}) = 0.$$

The subscript i refers to the region of interest and η_i is the appropriate similarity variable. This procedure has the advantage of producing the required similarity form directly. The expansions in each region must be substituted into the appropriate form of (1). Then terms of similar order in $\alpha_n(x/\epsilon)$ must be grouped so as to develop a sequence of ordinary differential equations for the $f_{ni}(\eta)$. If one has a good idea of the α_n sequence initially then the procedure can be carried out systematically.† In the present problem the sequence is not apparent at the outset. (And one may examine the final similarity solutions in § 5 to see that guessing the proper forms might be difficult.) As a result, the grouping process becomes difficult because of the presence of derivatives of unknown α_{ni} functions. If the possibility of logarithmic modifiers to algebraic orders of x/ϵ is recognized, then it becomes difficult to proceed systematically. Rather one would be inclined to use a trial-and-error procedure in which an α_{ni} is assumed so that explicit equations for the analogous f_{ni} can be obtained. Then matching could be carried out and the results examined for consistency. The presence of unmatchable terms would imply that alterations in the expansions were necessary. It would appear that this procedure is rather cumbersome.

The forthcoming process of analysis and the solutions in § 5 indicate the source of the complexity in the present problem. Unlike the problem solved by Klemp & Acrivos (1972) the eigenfunction here affects the expansions almost from the outset. One may observe the resulting logarithmic effect in the lowest order term for the wall shear. In retrospect then, it may be observed that the source of difficulty is the unexpectedly strong influence of eigenfunctions on the solution.

An alternative to the procedure described above involves the development of limit-process expansions based on the artificial parameter ϵ; see Lagerstrom & Cole (1955), Chang (1961), Van Dyke (1964) and Cole (1968). Solutions to (1) are sought in the form

$$g(x, y; \epsilon) \sim \sum_{n=0}^{\infty} \tilde{\alpha}_{ni}(\epsilon) \tilde{f}_{ni}(x, y_i), \quad \lim_{\epsilon \to 0} (\tilde{\alpha}_{(n+1)i}/\tilde{\alpha}_{ni}) = 0.$$

The subscript i has the same meaning as above and y_i is the appropriately stretched transverse variable. The expansion is defined for the limit process $\epsilon \to 0$, x and y_i fixed. When this expansion is substituted into the appropriate form of (1) it is relatively easy to form unambiguous groupings in the unknown sequence.‡ This results in partial differential equations for the functions $\tilde{f}_{ni}$. One can then obtain appropriate formal solutions and carry out the necessary matching procedure. A sequence of the latter operations will lead to specific definitions for the $\tilde{\alpha}_{ni}$. There will remain after each step undefined functions of x in the $\tilde{f}_{ni}$. These are obtained by writing the now known $\tilde{\alpha}_{ni}(\epsilon)$ in the form $\tilde{\alpha}_{ni}[(\epsilon/x)\,x]$, combining them with the appropriate $\tilde{f}_{ni}$ and insisting that the solution be independent of explicit functions of x. Although these operations are algebraically tedious, they do permit a systematic derivation of the solution with a minimum of trial-and-error iterations. It is essentially this property which makes the artificial-parameter development useful in this problem.

One can certainly present arguments for using one form of development or the other. For some, the presence of the arbitrary length scale in the second

† Klemp & Acrivos (1972) develop co-ordinate expansion solutions which are essentially power series in $(x/\epsilon)^{-\frac{1}{3}}$. This form is suggested directly by the nature of the wall-layer equation. Only in the fourth-order term does the presence of an eigenfunction cause a logarithmic modifier.

‡ Of course one must ascertain when the square of a larger term is equivalent in magnitude to some smaller term and so forth. This complication arises in the present problem and is dealt with in a systematic way.

procedure may make the trial-and-error calculation more attractive. For others the systematic derivation permitted by the limit-process expansion development is more appealing. In any event the procedures are ultimately equivalent, so that the choice of method is, to some extent, immaterial.

Perhaps it is worthwhile to note here that the solutions in limit-process expansion form are formally less complex than the final co-ordinate expansions. This occurs because the $\tilde{\alpha}_{ni}$ sequence is defined by transcendental expressions which must themselves be written as asymptotic expansions in the process of developing the co-ordinate expansion. A related phenomenon was discussed by Kassoy (1970, 1973).

4. Solution

In §2 the classical boundary-layer approach was shown to be valid for the injection constant $C < C_0$, while the newer weak interaction theory was limited to $C > C_0$. For C asymptotically close to C_0, the assumptions upon which the analyses were based became invalid. However, it should be observed that in both cases the basic shear-layer solution is the same. It is reasonable to expect, therefore, that, as the injection constant passes through the critical value, the shear layer will continue to be described by Lock's solution. If we inject mass exactly at the critical rate C_0, it would appear that, in the neighbourhood of the wall, the first approximation to the stream function must be $\psi_0 = -\epsilon^{\frac{1}{2}} C_0 (2x)^{\frac{1}{2}}$. This result follows from the fact that in the inner part of the free shear layer

$$\psi \sim \epsilon^{\frac{1}{2}}[-(2x)^{\frac{1}{2}} C_0],$$

which must match with the wall-layer solution. To lowest order then, the mass injected from the wall is totally entrained by the free shear layer. This lowest order solution is deficient in several respects. It provides no quantitative description of the location of the dividing streamline with respect to the wall, and no indication of a pressure interaction appears. At this point, one is left in a bit of a quandary regarding the proper length scales to use in formulating the theory. The classical boundary-layer analysis of Kassoy (1970) suggests that, as $C \rightarrow C_0 -$, the dividing streamline is located at a distance from the wall which is larger than $O(Re^{-\frac{1}{2}})$ in magnitude. On the other hand, the interaction theory implies that, as $C \rightarrow C_0 +$, the distance measure is less than $O(Re^{-\frac{1}{3}})$. An application of limit-process analysis to (1 a) for length scales between $O(Re^{-\frac{1}{2}})$ and $O(Re^{-\frac{1}{3}})$ does not lead to a distinguished limit. Since it is clear that the wall-layer thickness in the present problem cannot be as large as $O(Re^{-\frac{1}{3}})$, we are left with the only other scaling that does lead to a distinguished limit: $Re^{-\frac{1}{2}}$. As we shall see, this is indeed the correct scaling for the wall layer. This seemingly peculiar result arises from a fundamental difference in the behaviour of the wall-layer solutions in the interaction theory and in the present work. In the former, the solution is truncated at a finite value of the wall-layer variable, while in the present analysis, the solution extends to asymptotically large values of the pertinent variable. This phenomenon leads to a shear-layer (or dividing-streamline) location which is slightly larger in magnitude than the thickness of the wall layer itself. This type of behaviour may be observed in Kassoy's (1970) blow-off description, where the boundary layer is formally scaled by $O(Re^{-\frac{1}{2}})$ but the dividing-streamline location is $O\{Re^{-\frac{1}{2}}\log[1/(C_0-C)]\}$ for $C\to C_0-$.

The analysis will be developed in terms of a wall layer $O(Re^{-\frac{1}{2}})$ in thickness, a free shear layer whose dimension is $O(Re^{-\frac{1}{2}})$ and the external flow. The lowest order solutions in each layer are $\psi = -\epsilon^{\frac{1}{2}}C_0(2x)^{\frac{1}{2}}$, $\epsilon^{\frac{1}{2}}(2x)^{\frac{1}{2}}F_0(\eta)$ and y respectively. Here, $F_0(\eta)$ is the similarity solution for Lock's mixing layer, η is the appropriate similarity variable and y represents the uniform stream.

4.1. *Wall layer*

The wall-layer variables are defined by the usual boundary-layer stretching transformations:

$$\hat{\psi} = \epsilon^{-\frac{1}{2}}\psi, \quad \hat{y} = \epsilon^{-\frac{1}{2}}y. \tag{2}$$

It follows from $(1a, c)$ and (2) that the wall-layer system may be written in the form

$$\hat{\psi}_{\hat{y}}\hat{\psi}_{x\hat{y}} - \hat{\psi}_x\hat{\psi}_{\hat{y}\hat{y}} = -p_x + \hat{\psi}_{\hat{y}\hat{y}\hat{y}} + \epsilon\hat{\psi}_{\hat{y}xx}, \tag{3a}$$

$$\epsilon[-\hat{\psi}_{\hat{y}}\hat{\psi}_{xx} + \hat{\psi}_x\hat{\psi}_{x\hat{y}}] = p_{\hat{y}} - \epsilon[\hat{\psi}_{x\hat{y}\hat{y}} + \epsilon\hat{\psi}_{xxx}], \tag{3b}$$

$$\hat{\psi}_{\hat{y}}(x,0) = 0, \quad \hat{\psi}(x,0) = -C_0(2x)^{\frac{1}{2}}, \tag{3c}$$

Here $p = (p' - p'_\infty)/\rho'_\infty U'^2_\infty$, where p'_∞ is the static pressure of the undisturbed field and ρ'_∞ the reference density. Since the external flow is basically a uniform stream, we expect the streamwise pressure gradient p_x to be an asymptotically small function determined by an interaction between the injectant layer and the external flow. The solution to (3) is sought in the form

$$\hat{\psi} \sim -C_0(2x)^{\frac{1}{2}} + \sum_{n=1}^{\infty}\mu_n(\epsilon)\hat{\psi}_n(x,\hat{y}), \tag{4a}$$

$$p \sim \sum_{n=1}^{\infty}\gamma_n(\epsilon)p_n(x,\hat{y}), \tag{4b}$$

where $\{\mu_n\}$ and $\{\gamma_n\}$ are asymptotic sequences which must be determined. The expansions are based on the limit process $\epsilon\to 0$, x and $\hat{y}$ fixed. Substitution of $(4a)$ into $(3b)$ implies that $p_{\hat{y}} = O(\epsilon\mu_1(\epsilon))$. Hence, for $n < n^*$, where $\gamma_{n^*} = O(\epsilon\mu_1)$, $p_n = p_n(x)$. For the calculation to follow the magnitude of the lateral pressure gradient is negligible. A matching condition with the shear layer must be specified to complete the system.

4.2. *Shear layer*

The free-shear-layer equation may be obtained from $(1a)$ by using the transformations

$$\bar{\psi} = \epsilon^{-\frac{1}{2}}\psi, \quad Y = [y - g(x;\epsilon)]\,\epsilon^{-\frac{1}{2}}, \tag{5}$$

where $Y = 0$ is defined by $\bar{\psi}\,(x,0) = 0$. The corresponding description of the dividing streamline is assumed to be

$$y_D = g(x,\epsilon) \sim \sum_{n=1}^{\infty}\gamma_n(\epsilon)g_n(x). \tag{6}$$

The asymptotic sequence is the same as that used in the pressure expansion (4b), since the pressure interaction is in essence controlled by the shape of the dividing streamline. Since the dividing streamline is located at a distance from the wall which is larger than a classical boundary-layer thickness, it can be asserted that $\epsilon^{\frac{1}{2}} \ll \gamma_1 \ll 1$.

The shear-layer equation may be written in the form

$$\overline{\psi}_Y \overline{\psi}_{xY} - \overline{\psi}_x \overline{\psi}_{YY} = -p_x + \overline{\psi}_{YYY} + g'^2 \overline{\psi}_{YYY} + 2g' \epsilon^{\frac{1}{2}} \overline{\psi}_{YYx} - g'' \epsilon^{\frac{1}{2}} \overline{\psi}_{YY} + \epsilon \overline{\psi}_{Yxx}. \tag{7}$$

The pressure gradient is defined by (4b). Primes in (7) denote differentiation with respect to x. The third, fourth and fifth terms on the right-hand side represent effects of longitudinal curvature. Solutions to (7) must satisfy the dividing-streamline boundary condition

$$\overline{\psi}(x, Y = 0) = 0 \tag{8}$$

and appropriate matching conditions with the wall and external-flow layers. The former matching condition may be developed by applying the wall-layer limit process $\epsilon \to 0$, $\hat{y}$ fixed, to (5) in the inside region of the shear layer where $y < g(x; \epsilon)$. Thus, we find

$$\overline{\psi}(x, Y \to -\infty) = \hat{\psi}(x, \hat{y} \to \infty). \tag{9}$$

An application of the Euler limit process $\epsilon \to 0$, y fixed, to (5) suggests that the matching condition with the external flow has the familiar form

$$\epsilon^{\frac{1}{2}} \overline{\psi}(x, Y \to \infty) = \psi(x, y \to 0). \tag{10}$$

The stream function is assumed to have the form

$$\overline{\psi}(x, Y; \epsilon) \sim (2x)^{\frac{1}{2}} [F_0(\eta) + \sum_{n=1}^{\infty} \nu_n(\epsilon) F_n(x, \eta)], \tag{11}$$

where $F_0(\eta)$ is Lock's mixing-layer similarity solution, $\eta = Y(2x)^{-\frac{1}{2}}$ and $\{\nu_n(\epsilon)\}$ is an asymptotic sequence defined for the limit process $\epsilon \to 0$, x and Y fixed. The lowest order term in (11) may be observed to match with the basic wall-layer solution and the uniform stream.

4.3. *External flow*

An application of the Euler limit $\epsilon \to 0$, x and y fixed, to (1a) yields the Euler equation subject to the uniform-flow boundary condition. It follows that the external stream function may be written in the form

$$\psi(x, y; \epsilon) \sim y + \sum_{n=1}^{\infty} \sigma_n(\epsilon) \psi_n(x, y), \tag{12}$$

where $\{\sigma_n(\epsilon)\}$ is an asymptotic sequence. The functions $\psi_n(x, y)$, described by $\nabla^2 \psi_n = 0$ for the cases of interest here, represent corrections to the uniform flow due to the presence of the structure in the two internal layers. The boundary conditions on ψ_n may be obtained by an explicit development of (10). Thus, combining (5), (10)–(12), the definition of η, and the asymptotic behaviour

$$F_0(\eta \to \infty) \sim \eta - K,$$

where $K = 0{\cdot}38$, we find that

$$\sum_{n=1}^{\infty} \sigma_n(\epsilon)\,\psi_n(x, y \to 0) \sim -g(x;\epsilon) - K(2x)^{\frac{1}{2}}\epsilon^{\frac{1}{2}} + O(\epsilon^{\frac{1}{2}}\nu_1(\epsilon) f_1(x, \eta \to \infty)).$$

It follows from this result and (6) that

$$\left.\begin{aligned} \sigma_n &= \gamma_n, \quad \psi_n(x,0) = -g_n(x), \quad O(\gamma_n) \gg \epsilon^{\frac{1}{2}} && \text{for } n < m, \\ \sigma_m &= \gamma_m = \epsilon^{\frac{1}{2}}, \quad \psi_m(x,0) = -g_m(x) - K(2x)^{\frac{1}{2}} && \text{for } n = m. \end{aligned}\right\} \tag{13}$$

Classical slender-body theory may be used to show that the pressure interactions are given by

$$p_n(x) = -\pi^{-1}\int_0^{\infty} \frac{g_n'(\xi)\,d\xi}{x-\xi}. \tag{14}$$

The specific functional dependence of $g_n(x)$ will be obtained in the analysis to follow.

4.4. *Higher order solutions*

If (4) is substituted into ($3a, c$), we obtain the system describing the first correction to the stream function:

$$\left[\frac{\partial^3}{\partial \hat{y}^3} + C_0(2x)^{-\frac{1}{2}}\frac{\partial^2}{\partial \hat{y}^2}\right]\hat{\psi}_1 = \frac{\gamma_1}{\mu_1} p_{1x},$$

$$\hat{\psi}_{1\hat{y}} = \hat{\psi}_1 = 0, \quad \hat{y} = 0.$$

The pressure term is included for generality since the ratio $\gamma_1/\mu_1 \leqslant O(1)$ is unknown at present. A general solution is

$$\left.\begin{aligned} \hat{\psi}_1 &= a_1(x)[\exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1] - \frac{\gamma_1}{\mu_1}\frac{(2x)^{\frac{3}{2}}}{2C_0} p_{1x}\hat{\eta}^2, \\ \hat{\eta} &= \hat{y}/(2x)^{\frac{1}{2}}. \end{aligned}\right\} \tag{15}$$

The form necessary for matching with the shear layer can be obtained from ($4a$), (9), (11), (15) and the asymptotic mixing-layer behaviour (Kassoy 1971)

$$F_0(\eta \to -\infty) \sim -C_0 + k\exp(C_0\eta) + \dots \quad (k = 1{\cdot}1502).$$

Thus

$$(2x)^{\frac{1}{2}} C_0 + \mu_1(\epsilon)\,a_1(x)\exp(C_0\hat{\eta}) + \dots \sim (2x)^{\frac{1}{2}}[-C_0 + k\exp(C_0\eta + \dots)]. \tag{16}$$

Matching is completed by writing η in terms of $\hat{\eta}$ and ϵ. We find

$$\eta = \hat{\eta} - \epsilon^{-\frac{1}{2}}(2x)^{-\frac{1}{2}} g(x;\epsilon), \tag{17}$$

where $g(x;\epsilon)$ is given by (6). The relevant form of the matching condition, found by substituting (17) into (16) and cancelling the leading terms, is

$$\mu_1 a_1(x) = k(2x)^{\frac{1}{2}}\exp[-(2x)^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}C_0(\gamma_1 g_1 + \gamma_2 g_2 + \dots)]. \tag{18}$$

This equation is satisfied to lowest order if

$$\gamma_1 = \epsilon^{\frac{1}{2}}\log(1/\mu_1), \quad g_1 = (2x)^{\frac{1}{2}} C_0^{-1}, \tag{19a}$$

$$\gamma_2 = \epsilon^{\frac{1}{2}}, \quad g_2 = \frac{(2x)^{\frac{1}{2}}}{C_0}\log[k(2x)^{\frac{1}{2}} a_1^{-1}(x)]. \tag{19b}$$

These results indicate that the dividing streamline is to a first approximation located at a distance from the wall slightly larger (by the factor $\log(1/\mu_1)$) than the classical boundary-layer thickness. However, since the basic shape is parabolic, no pressure interaction develops; $p_1 = 0$. Hence, the first pressure-gradient effect can arise at best from the slightly smaller second term $\gamma_2 g_2 = O(\epsilon^{\frac{1}{2}})$, presuming, of course, that $a_1(x)$ is not proportional to $(2x)^{\frac{1}{2}}$. This function and the parameter $\mu_1(Re)$ remain to be found. An examination of higher order stream-function equations derived from (3a) and (4a) suggests that we may write

$$\hat{\psi} \sim -C_0(2x)^{\frac{1}{2}} + \mu_1 a_1(\exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1) + \epsilon^{\frac{1}{2}}[a_2(\exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1) - (2C_0)^{-1}(2x)^{\frac{3}{2}} p_{2x}\hat{\eta}^2] + o(\epsilon^{\frac{1}{2}}). \quad (20)$$

If $\mu_1 = \epsilon^{\frac{1}{2}}$, then we set $a_1 = 0$ and use only the last term in (20). The form of $p_2(x)$ can be found from (14) and (19b) once a_1 has been evaluated.

The choice of μ_1 cannot be made by any further manipulations in the wall layer. In fact, since only the first exponential term in (20) has been matched, it is apparent that the selection follows from matching the algebraic terms including the pressure interaction with corrections to Lock's solution in the free shear layer. Of course, recognizing now that the wall-layer solution is a kind of linearized boundary layer, one might be tempted to choose $\mu_1 = \epsilon^{\frac{1}{2}}$, the usual order of correction to the classical boundary layer. However, this choice does not lead to satisfaction of the matching condition.

Rather than carrying out the remaining analysis for a general μ_1, we shall here specify the result

$$\mu_1 = \epsilon^{\frac{1}{2}} \log(1/\mu_1) \quad (21)$$

and show that the choice satisfies all required conditions. This form of presentation is chosen for pedagogical purposes because it permits a rather more concise formulation than does the general calculation. It may be noted that the logarithmic factor is necessary because of the presence of eigenfunction solutions in the shear layer.

Once (21) is inserted into (20), a consideration of the similarity requirements can be made. Thus, $\hat{\psi}(2x)^{-\frac{1}{2}} = \psi'(2\nu' U'_\infty x')^{-\frac{1}{2}}$ must be independent of the artificial length scale L'. The definition of $\hat{\eta} = \hat{y}(2x)^{-\frac{1}{2}}$ implies that it is effectively a similarity variable. Hence, we need be concerned with only μ_1, a_1, a_2 and the group $(2x)^{\frac{3}{2}} p_{2x}$. The necessary conditions may be obtained by writing

$$\epsilon = Re^{-1} = x(Re\, x)^{-1}$$

(where the term in parentheses is independent of L'), inserting this form of ϵ into (20) and (21) and regrouping terms appropriately. This calculation appears in the appendix. Thus, the similarity requirement is met to $O(\epsilon^{\frac{1}{2}})$ if

$$a_1 = \text{constant}, \quad a_2 = \tfrac{1}{2}a_1 \log x + a_2^*, \quad (2x)^{\frac{3}{2}} p_2' = \text{constant}, \quad (22)$$

where a_2^* is a constant. The quantity p_2 can be calculated from (14), (19b) and the first part of (22). An appropriate evaluation gives

$$p_2(x) = \pi(2C_0)^{-1}(2x)^{-\frac{1}{2}}, \quad (23a)$$

or

$$p_2'(x) = -\pi(2C_0)^{-1}(2x)^{-\frac{3}{2}}. \quad (23b)$$

The latter form of the induced favourable pressure gradient is consistent with the last condition in (22).

The asymptotic sequence for the shear-layer expansion in (11) can be inferred from the matching form of the wall-layer stream-function expansion. In order to construct this expression, we must first combine (6), (17) and (19) into

$$C_0\hat{\eta} = -\log\mu_1 + C_0\eta + \Omega(x) + \epsilon^{-\frac{1}{2}}\gamma_3 C_0 g_3(2x)^{-\frac{1}{2}} + \dots, \tag{24}$$

where $\Omega(x) = \frac{1}{2}\log(2k^2x/a_1^2)$. Similarity requirements imply that

$$\gamma_n = \epsilon^{\frac{1}{2}}\log^{2-n}(1/\mu_1), \quad g_n = O(x^{\frac{1}{2}}\log^{n-2}x) \quad (n = 3, 4, 5, \dots),$$

which will be verified in the forthcoming matching for $n = 3$ and 4. It follows from consideration of (3), the forms of expansions suggested above, and similarity arguments like those in the appendix that (20) can be generalized to

$$\hat{\psi} \sim -C_0(2x)^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\log(1/\mu_1)\sum_{n=1}^{\infty}[a_n(\exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1)$$
$$-(2C_0)^{-1}(2x)^{\frac{3}{2}}p_{nx}\hat{\eta}^2]\log^{1-n}(1/\mu_1) + O(\epsilon\log^2\mu_1),$$

where $p_{1x} = 0$, p_{2x} is given in (23) and a_1 and a_2 are given in (22). The form of a_3 is also required to develop the matching expression to the desired order. As shown in (A 6*c*), $a_3 = -\frac{1}{2}a_1\log x + a_3^*$, where a_3^* is a constant. Then, substituting (21) and (24) into the generalized $\hat{\psi}$, regrouping terms in the shear-layer form and invoking the matching condition in (9), we find

$$\begin{aligned}\hat{\psi}(x,\hat{\eta}\to\infty) &\sim \overline{\psi}(x,\eta\to-\infty)\\
&\sim -C_0(2x)^{\frac{1}{2}} + k(2x)^{-\frac{1}{2}}e^{C_0\eta}[1 - \log^{-1}\mu_1(\bar{g}_3 + a_2a_1^{-1})\\
&\quad + \log^{-2}\mu_1(\bar{g}_4 + \tfrac{1}{2}\bar{g}_3^2 + a_2a_1^{-1}\bar{g}_3 + a_3a_1^{-1}) + O(\log^{-3}\mu_1)]\\
&\quad + \epsilon^{-\frac{1}{2}}\log^2\mu_1[C_0^{-2}\beta_2 - a_1 + o(1)]\\
&\quad - \epsilon^{-\frac{1}{2}}\log\nu_1[(C_0\eta + \Omega)(2C_0^{-2}\beta_2 - a_1) - a_1 - a_2 + C_0^{-2}\beta_3 + o(1)]\\
&\quad + \epsilon^{-\frac{1}{2}}[(C_0\eta + \Omega)^2C_0^{-2}\beta_2 + (C_0\eta+\Omega)(2C_0^{-2}\beta_3 - a_2) + \bar{g}_3(2C_0^{-2}\beta_2 - a_1)\\
&\quad - a_2 - a_3 + C_0^{-2}\beta_4 + o(1)] + O(\epsilon^{-\frac{1}{2}}\log^{-1}\mu_1),\end{aligned} \tag{25}$$

where

$$\beta_n = -(2x)^{\frac{3}{2}}p_n'(x)(2C_0)^{-1}, \quad \bar{g}_n = (2x)^{-\frac{1}{2}}C_0g_n, \quad \Omega = \tfrac{1}{2}\log(2k^2x/a_1^2). \tag{26}$$

The largest explicit corrections in (25) arise from the sequence $\{O(\log^{-n}\mu_1)\}$, $n = 0, 1, 2, \dots$. If one assumes in (11) that $\nu_1(\epsilon) = \log^{-1}(1/\mu_1)$, an inconsistency will arise for the resulting sequence of $F_n(x,\eta)$ equations. This difficulty implies that the $O(\log^{-1}\mu_1)$ sequence must be annihilated since it cannot be matched. This can be accomplished if

$$\bar{g}_3 = -a_2a_1^{-1} = -\tfrac{1}{2}\log x - a_2^*a_1^{-1}, \tag{27a}$$

$$\bar{g}_4 = -\tfrac{1}{2}\bar{g}_3^{-2} - a_2a_1^{-1}\bar{g}_3 - a_3a_1^{-1} = \tfrac{1}{8}\log^2 x + \tfrac{1}{2}(a_2^*a_1^{-1} + 1)\log x + \tfrac{1}{2}a_2^{*2}a_1^{-2} - a_3^*a_1^{-1}, \tag{27b}$$

where a_1, a_2 and a_3 have been defined previously. These results verify the form of g_n found from similarity requirements mentioned just below (24). The higher order pressure interactions, found from (14), (26) and (27), are

$$p_3(x) = -\pi(2C_0)^{-1}(2x)^{-\frac{1}{2}},$$

$$p_4(x) = [\tfrac{1}{4}\pi C_0^{-1}\log x - \pi C_0^{-1}(1+\tfrac{1}{2}a_2^* a_1^{-1})](2x)^{-\frac{1}{2}}.$$

It is to be noted that the pressure interaction is fully specified except for the ratio a_2^*/a_1.

The remaining terms in (25) suggest that the first three terms of the asymptotic sequence in the shear layer are $\nu_n(\epsilon) = \epsilon^{-\frac{1}{2}}\log^{3-n}(1/\mu_1)$, $n = 1, 2, 3$. Hence, (11) may be substituted into (7) (where use is made of (4*b*), (19) and (23) to find the pressure contribution) to find the first three describing equations:

$$L(F_i) = 2x(F_0' F_{i\eta x} - F_0'' F_{ix}), \quad L = \frac{\partial^3}{\partial\eta^3} + F_0\frac{\partial^2}{\partial\eta^2} + F_0'' \quad (i = 1, 2), \qquad (28a)$$

$$L(F_3) = 2x(F_0' F_{3\eta x} - F_0'' F_{3x}) + 2xp_2'(x). \qquad (28b)$$

The solutions $F_n(x,\eta)$ must satisfy the dividing-streamline boundary condition $F_n(x,0) = 0$, derived from (8), a matching condition for $\eta \to -\infty$, which can be ascertained (25), and a matching condition with the external field. This last condition is found by writing (10) in the streamwise velocity form

$$\overline{\psi}_Y(x, Y\to\infty) = \psi_y(x, y \to 0),$$

substituting the appropriate expansions and using the relation

$$y = \epsilon^{\frac{1}{2}}(-g_1\log\mu_1 + Y + g_2 - g_3\log^{-1}\mu_1 + \ldots),$$

where g_1, g_2 and g_3 are specified in (19) and (27). It follows after suitable algebraic manipulation that

$$\overline{\psi}_Y(x, Y\to\infty) \sim 1 - \epsilon^{\frac{1}{2}}\log\mu_1\psi_{1y}(x,0) + \epsilon^{\frac{1}{2}}\psi_{2y}(x,0) + \ldots. \qquad (29)$$

The Bernoulli equation $p + \frac{1}{2}(\psi_y^2 + \psi_x^2) = \frac{1}{2}$ can be used to relate the stream-function derivatives at $y = 0$ to the previously derived interaction pressure gradients. We find that

$$\psi_{1y}(x,0) = -p_1(x) = 0, \quad \psi_{2y}(x,0) = -p_2(x) = -2C_0\beta_2(2x)^{-\frac{1}{2}}. \qquad (30)$$

Then a comparison of the y derivative of (11) with the form developed from (29) and (30) indicates that for $\eta = Y(2x)^{-\frac{1}{2}} \to \infty$

$$F_0' = 1, \quad F_{1\eta} = F_{2\eta} = 0, \quad F_{3\eta} = -2C_0\beta_2(2x)^{-\frac{1}{2}}. \qquad (31)$$

Solutions for the F_n systems can now be developed. A careful examination of the (x,η) dependence of the third, fourth and fifth terms in (25) (which provide conditions on $F_i(x, \eta \to -\infty)$) indicates that these functions should be written in the form

$$F_1 = (2x)^{-\frac{1}{2}}f_{10}(\eta), \quad F_2 = (2x)^{-\frac{1}{2}}[f_{20}(\eta) + (\log x)f_{21}(\eta)], \qquad (32a, b)$$

$$F_3 = (2x)^{-\frac{1}{2}}[f_{30}(\eta) + (\log x)f_{31}(\eta) + (\log^2 x)f_{32}(\eta)]. \qquad (32c)$$

Similarity requirements imply that $\bar{\psi}(2x)^{-\frac{1}{2}} \equiv \psi'(2x'\nu'U'_\infty)^{-\frac{1}{2}}$ must be independent of L'. If the procedure used in the appendix is applied to the combination of (11) and (32), it follows that

$$f_{21} = 4f_{32} = f_{10}, \quad f_{31} = \tfrac{1}{2}f_{20} - f_{10}. \tag{33}$$

Hence, we need to find only the three functions f_{i0}, $i = 1$–3.

The system describing f_{10} can be found from (25), (28*a*), (31), (32*a*), the required definitions therein and the dividing-streamline condition. Thus

$$f'''_{10} + F_0 f''_{10} + F'_0 f'_{10} = 0,$$

$$f'_{10}(\eta \to \infty) = f_{10}(0) = 0, \quad f_{10}(\eta \to -\infty) = C_0^{-2}\beta_2 - a_1.$$

This describes an 'eigenvalue' problem for $f'_{10}(\eta)$. The solution may be written simply as

$$f_{10} = (C_0^{-2}\beta_2 - a_1)(1 - F'_0/F'_0(0)). \tag{34}$$

Similar manipulations for F_2 verify that the condition on f_{21} in (33) is satisfied identically. It follows that f_{20} is described by

$$f'''_{20} + F_0 f''_{20} + F'_0 f'_{20} = -2(C_0^{-2}\beta_2 - a_1)F''_0,$$

$$f'_{20}(\eta \to \infty) = f_{20}(0) = 0,$$

$$f_{20}(\eta \to -\infty) = (2C_0^{-2}\beta_2 - a_1)C_0\eta + (2C_0^{-2}\beta_2 - a_1)\hat{\Omega} - a_1 - a_2^* + C_0^{-2}\beta_3,$$

where $\hat{\Omega} = \frac{1}{2}\log(2k^2a_1^{-2})$. A first integral of the equation is

$$f''_{20} + F_0 f'_{20} = -2(C_0^{-2}\beta_2 - a_1)F'_0 + \Gamma_2. \tag{35}$$

The integration constant Γ_2 and the quantity a_1 may now be found by evaluating (35) at $\pm\infty$. This results in two equations for the unknowns which lead to

$$\left.\begin{aligned} \Gamma_2 &= -2\beta_2(C_0^2+2)^{-1} = -0{\cdot}745, \\ a_1 &= 2\beta_2(C_0^2+1)C_0^{-2}(C_0^2+2)^{-1} = 1{\cdot}71. \end{aligned}\right\} \tag{36}$$

It follows that $\hat{\Omega} = \frac{1}{2}\log(2k^2a_1^{-2}) = -0{\cdot}049$. The solution for $f'_{20}(\eta) \equiv v$ can be found formally by a numerical integration of the completely specified system

$$v'' + F_0 v' + F'_0 v = -2(\beta_2 C_0^{-2} - a_1)F''_0,$$

$$v(\infty) = 0, \quad v(-\infty) = (2\beta_2 C_0^{-2} - a_1)C_0.$$

However, the solution is not completely specified because the 'eigenfunction' $F''_0(\eta)$ satisfies the equation and boundary conditions exactly. Hence, $f_{20}(\eta)$ must be written generally as

$$f_{20}(\eta) = \int_0^\eta v(\tau)\,d\tau + K_{20}(1 - F'_0/F'_0(0)),$$

where K_{20} is an unknown constant. When the full boundary condition for $\eta \to -\infty$ is satisfied a single equation for the two unknowns K_{20} and a_2^* results. This indeterminacy arises essentially from the first eigenfunction of Lock's free-shear-layer solution (in a manner analogous to that in the Blasius problem (Van Dyke 1964)). The value of a_2^* is indeterminate since it depends (in this purely similarity problem) on the flow properties near the leading edge of the plate, which are not well defined. If the flow were started in a more realistic fashion and only asymptotically approached the present similarity solution, then a_2^* could be determined from the requisite initial data. A related indeterminacy was found by Klemp & Acrivos (1972) in their higher order interaction analysis.

The F_3 solution is considered in the same manner. The f_{32} and f_{31} systems have solutions which verify (33). Finally

$$f_{30}''' + F_0 f_{30}'' - F_0' f_{30}' = -2C_0\beta_2 + F_0' f_{20}' - F_0'' f_{20},$$

$$f_{30}'(\eta \to \infty) = -2C_0\beta_2, \quad f_{30}(0) = 0,$$

$$f_{30}(\eta \to -\infty) = \beta_2\eta^2 + C_0\eta[2\beta_2 C_0^{-2}\hat{\Omega} + 3\beta_3 C_0^{-2} - a_2^*] + O(1).$$

This system can be analysed in the manner used to consider f_{20}. It is to be noted, however, that the system for $f_{30}'(\eta)$ is not completely specified owing to the presence of the unknown a_2^* in the $O(\eta)$ term of the last condition.

It should be noted that the choice of μ_1 in (21) has been validated *a posteriori*. Any other choice (i.e. $\mu_1 = \epsilon^{\frac{1}{2}}$) leads to an overspecified system for the choice of Γ_2 and a_1.

5. Results

The formulae for the stream function $\hat{\psi}$, the dividing-streamline location y_D and the interaction pressure can now be written in parameter form. Thus from the appendix and (21), (23) and (36) we obtain

$$\hat{\psi} = -C_0(2x)^{\frac{1}{2}} + \epsilon^{\frac{1}{2}}\log(1/\mu_1)\left[m(\hat{\eta})\sum_{n=1}^{\infty} a_n \log^{1-n}(1/\mu_1) - (2C_0)^{-1}\hat{\eta}^2(2x)^{\frac{3}{2}}\sum_{n=2}^{\infty} p_n'(x)\log^{1-n}(1/\mu_1)\right] + O_l(\epsilon),$$

where
$$\mu_1 = -\epsilon^{\frac{1}{2}}\log\mu_1, \quad m(\hat{\eta}) = \exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1,$$

$$a_1 = 1\cdot71\ldots, \quad a_2 = \tfrac{1}{2}a_1\log x + a_2^*, \quad a_3 = -\tfrac{1}{2}a_1\log x + a_3^*,$$

$$a_4 = \tfrac{1}{4}a_1\log^2 x + \tfrac{1}{2}(a_1 - a_3^*)\log x + a_3^*\log 2 + a_4^*,$$

$$p_2 = (2C_0)^{-1}\pi(2x)^{-\frac{1}{2}}, \quad p_3 = -(2C_0)^{-1}\pi(2x)^{-\frac{1}{2}}, \qquad (37a, b)$$

$$p_4 = [(4C_0)^{-1}\pi\log x - (2a_1C_0)^{-1}\pi(a_2^* + 2a_1)](2x)^{-\frac{1}{2}}. \qquad (37c)$$

Here the starred constants are indeterminate for the reasons discussed previously. The symbol $O_l(\epsilon)$ indicates an algebraic dependence $O(\epsilon)$ with logarithmic modifiers due to $\log\mu_1$. The dividing streamline is described by

$$y_D = C_0^{-1}(2x)^{\frac{1}{2}}\epsilon^{\frac{1}{2}}\log(1/\mu_1)\left[1 + \sum_{n=2}^{\infty}(g_n/g_1)\log^{1-n}(1/\mu_1)\right] + O_l(\epsilon),$$

$$g_1 = C_0^{-1}(2x)^{\frac{1}{2}}, \quad g_2 = g_1\log(ka_1^{-1}(2x)^{\frac{1}{2}}), \quad k = 1\cdot1502,$$

$$g_3 = -(2a_1)^{-1}g_1(a_1\log x + 2a_2^*),$$

$$g_4 = g_1[\tfrac{1}{8}\log^2 x + \tfrac{1}{2}((a_2^*/a_1) + 1)\log x + a_2^{*2}/2a_1^2 - a_3^*/a_1],$$

which results from a combination of (6), (19), (24), (26) and (27). Finally the interaction pressure field expansion, found from (4), has the form

$$p = \epsilon^{\frac{1}{2}} \sum_{n=2}^{\infty} p_n(x) \log^{2-n}(1/\mu_1) + O_l(\epsilon),$$

where the first three $p_n(x)$ are given in (37).

The similarity form of these expansions can now be obtained by using (A 3) and (A 4) for $\log(1/\mu_1)$. After appropriate regrouping of terms it is found that the dimensional wall shear stress distribution developed from the stream-function formula is

$$\tau'_w = \frac{\mu'_\infty U'_\infty}{x'} \frac{C_0^2 a_1}{4} T \left[1 - 2\frac{\log T}{T} + \frac{2}{T}\left(\frac{a_2^*}{a_1} + \log 2 + \frac{\pi}{2C_0^4 a_1}\right) + \frac{4 \log T}{T^2} + \frac{2}{T^2}\left(\frac{2a_3^*}{a_1} - 2\log 2 - \frac{\pi}{C_0^4 a_1}\right) + O(T^{-3}\log^2 T)\right],$$

where $T = \log(Re\,x)$ is independent of L' as required by similarity. The dimensional dividing streamline is described by

$$y'_D = \left(\frac{2x'\nu'_\infty}{U'_\infty}\right)^{\frac{1}{2}} \frac{T}{2C_0}\left[1 - 2\frac{\log T}{T} + \frac{2}{T}\log\left(\frac{2^{\frac{3}{2}}k}{a_1}\right) + 4\frac{\log T}{T^2} - \frac{4}{T^2}\left(\frac{a_2^*}{} + \log 2\right) + O(T^{-3}\log^2 T)\right].$$

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Nonlinear streaming effects associated with oscillating cylinders

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By A. BERTELSEN,

$;T)\Big].$

A comparison of the wall shear stress magnitude for classical boundary-layer theory, $\tau'_w = O([Re\,x]^{-\frac{1}{2}})$, and the interaction theory, $\tau'_w = O([Re\,x]^{-\frac{5}{6}})$, with that in the present analysis, $\tau'_w = O([Re\,x]^{-1}\log x')$, shows that the implied minimum in the wall shear distribution discussed by Klemp & Acrivos (1972) is real. We may now observe that the wall shear is finite, though relatively small at the critical injection rate. The basic approximation to the wall shear, arising from the second term in (20), is observed to be purely viscous in origin. It is only in the slightly smaller correction that the interaction pressure force makes a contribution. Hence, we see that, at the critical injection rate, the interaction is still relatively weak. Presumably, the latter effect becomes of the same order as the viscous effect for C very slightly greater than C_0. Of course, for C sufficiently greater than C_0, the pressure interaction dominates the flow in the wall region.

The dividing-streamline shape is that of a slightly perturbed parabola. One should note that the perturbation is absolutely essential. For an exactly parabolic displacement body will produce no pressure interaction at all. The non-monotonic variation in wall shear can now be seen to result from the way in which the pressure interaction develops. As C passes through C_0, the displacement body is perturbed slightly from the parabolic shape predicted by boundary-layer theory. The disturbance is smaller in magnitude than higher order viscous corrections to boundary-layer theory; hence the decreasing segment of the variation. Then when $C > C_0$, the displacement body is further distorted from the parabolic shape to an $x^{\frac{2}{3}}$ distribution. It follows that the interaction pressure is the dominating force near the wall and the wall shear increases again.

The philosophy used to develop this similarity solution for injection at exactly the critical rate would appear to be applicable to the problem of describing the blow-off regime on a flat plate with uniform injection of boundary-layer magnitude. The essential notion is that higher order viscous theory, including pressure interaction resulting from locally large streamline slopes, must be considered from the start. Presumably a point of exactly zero wall shear will be avoided, in analogy with the present problem, and the apparent singularity discussed by Catherall, Stewartson & Williams (1965) and Kassoy (1973) will be absent. Such a study should provide a transition between the upstream boundary-layer flow and the downstream interaction zone (Kassoy 1971; Klemp & Acrivos 1972).

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Appendix

The explicit asymptotic expression for $\log(1/\mu_1)$ can be found by expanding (21) in the limit $\epsilon \to 0$. The resulting form is

$$-\log\mu_1 = \beta - \log\beta + \beta^{-1}\log\beta + \beta^{-2}(\tfrac{1}{2}\log^2\beta - \log\beta) + O(\beta^{-3}\log^3\beta), \quad \text{(A 1)}$$

where

$$\beta = \tfrac{1}{2}\log(1/\epsilon). \quad \text{(A 2)}$$

Now if $\epsilon = Re^{-1} = x(Re\,x)^{-1}$ is substituted into (A 1) and (A 2) and an asymptotic expansion is developed in terms of the limit x fixed, $Re\,x \to \infty$ (implying a vanishingly small viscosity), then $\log(1/\mu_1)$ can be written as

$$\begin{aligned}-\log\mu_1 = \mathscr{F}(T) - \tfrac{1}{2}\log x + T^{-1}\log x + T^{-2}[(2\log x)\log T + \tfrac{1}{2}\log^2 x \\ - 2\log x(\log 2 + 1)] + O((\log x)\,T^{-3}\log^2 T),\end{aligned} \quad \text{(A 3)}$$

where

$$\begin{aligned}\mathscr{F}(T) = {} & \tfrac{1}{2}T - (\log T - \log 2) + 2T^{-1}(\log T - \log 2) \\ & + T^{-2}[2\log^2 T + 4\log T(-1 - \log 2) + 2\log^2 2 + 4\log 2] \\ & + O(T^{-3}\log^3 T).\end{aligned} \quad \text{(A 4)}$$

Here $T = \log(Re\,x)$ is a variable independent of the length scale L'.

We presuppose that (20) can be generalized to

$$\hat{\psi}(2x)^{-\frac{1}{2}} = -C_0 + (2Re\,x)^{-\frac{1}{2}}\Big[m(\hat{\eta})\sum_{n=1}^{\infty} a_n(x)\log^{2-n}(1/\mu_1) - (2C_0)^{-1}\hat{\eta}^2(2x)^{\frac{3}{2}}\sum_{n=2}^{\infty} p_n'(x)\log^{2-n}(1/\mu_1)\Big] + \ldots,$$

where $m(\hat{\eta}) = \exp(C_0\hat{\eta}) - C_0\hat{\eta} - 1$. (This is verified by subsequent developments in the main body of the work.) The similarity requirements can be developed by substituting (A 3) and (A 4) into the expansion and insisting that x appear only in the group $Re\,x$, which is independent of L'. It follows from the substitution that

$$\sum_{n=1}^{\infty} a_n(x)\log^{2-n}(1/\mu_1) = a_1\mathscr{F}(T)+a_2-\tfrac{1}{2}a_1\log x+T^{-1}(2a_3+a_1\log x)$$
$$+T^{-2}\log T\,(4a_3+2a_1\log x)$$
$$+T^{-2}\{4a_4+(-4\log 2+2\log x)\,a_3+(\tfrac{1}{2}\log^2 x$$
$$-2\log x\,(\log 2+1)]\,a_1\}+O(T^{-3}\log^2 T). \qquad \text{(A 5)}$$

Hence the L'-independent form results if

$$a_1 = \text{constant}, \qquad \text{(A 6}a\text{)}$$

$$a_2 = \tfrac{1}{2}a_1\log x+a_2^*, \quad a_2^* = \text{constant}, \qquad \text{(A 6}b\text{)}$$

$$a_3 = -\tfrac{1}{2}a_1\log x+a_3^*, \quad a_3^* = \text{constant}, \qquad \text{(A 6}c\text{)}$$

$$a_4 = \tfrac{1}{4}a_1\log^2 x+\tfrac{1}{2}(a_1-a_3^*)\log x+a_3^*\log 2+a_4^*, \quad a_4^* = \text{constant}, \qquad \text{(A 6}d\text{)}$$

where $a_i^* =$ constant, to be found. An analogous calculation must be done for the pressure-gradient summation in the $\hat{\psi}$ expression. This leads to the requirement

$$(2x)^{\frac{3}{2}}p_2'(x) = p_{20}^*, \quad (2x)^{\frac{3}{2}}p_3'(x) = p_{30}^*, \qquad \text{(A 7}a,b\text{)}$$

$$(2x)^{\frac{3}{2}}p_4'(x) = p_{40}^*+p_{41}^*\log x, \qquad \text{(A 7}c\text{)}$$

where $p_{ij}^* =$ constants to be found.

The results in (A 5)–(A 7) may be substituted into the $\hat{\psi}$ expression to find the required similarity solution. It remains to determine the constants in (A 6) and (A 7).

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